1 Prime numbers

1. According to the fundamental theorem of arithmetic, every integer may be written as a product of primes.

(a) Put the numbers 1,000,000, 1,000,004 and 999,999 in the form \(N = \prod_k \pi_k^{m_k}\) (you may use Matlab to find the prime factors).

(b) Give a generalized formula for the natural logarithm of a number \(N\) in terms of its primes \(\pi_k\).

2. Prime numbers may be identified using ‘sieves’

(a) By hand, perform the sieve Eratosthenes for \(n = 1\ldots49\). Circle each prime \(p\) then draw a slash through each number which is a multiple of \(p\).

(b) In part (a), which is the highest number you need to consider before all primes have been identified?

(c) Generalize: for \(n = 1\ldotsN\), which is the highest number you need to consider before all primes have been identified?

2 Greatest common divisors

Consider Euclid’s algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers

1. Understand Euclid’s algorithm

(a) Use the Matlab command `factor` to find the prime factors of \(a = 85\) and \(b = 15\). What is the greatest common prime factor of these two numbers?

(b) By hand, perform Euclid’s algorithm for \(a = 85\) and \(b = 15\).

(c) By hand, perform Euclid’s algorithm for \(a = 75\) and \(b = 25\). Is the result a prime number?

(d) Describe in your own words how the GCD algorithm works. Try the algorithm using numbers which have already been separated into factors (e.g. \(a = 5 \cdot 3\) and \(b = 7 \cdot 3\)).

2. Write a matlab function, `function x = my_gcd(a,b)` which uses Euclid’s algorithm to find the GCD of any two inputs \(a\) and \(b\). Test your function on the (a,b) combinations from parts (a) and (b). Include a printout (or handwrite) your algorithm to turn in.

Hints and advice:

- Don’t give your variables the same names as Matlab functions! Here, `gcd` is an existing function, so if you use it as a variable or function name, you won’t be able to use `gcd` to check your own function. Try `clear all` if you accidentally do this.
- Try using a ‘while’ loop for this exercise (see Matlab documentation for help).
- You may need to make some temporary variables for \(a\) and \(b\) in order to perform the algorithm.
3 Pythagorean triples

Euclid’s formula for the Pythagorean triples gives \( a = p^2 - q^2, \) \( b = 2pq, \) and \( c = p^2 + q^2. \)

1. What condition(s) must hold for \( p \) and \( q \) such that \( a, b, \) and \( c \) are always positive and nonzero?

2. Solve for \( p \) and \( q \) in terms of \( a, b, \) and \( c. \) Hint: you don’t need to use \( b.\)

3. Consider Figure 1.3 of Stillwell. Find \( p \) and \( q \) for the first five \((a,c)\) pairs in Plimpton 322.

4. Set \( n = p - q, \) and find a relationship between \( \sqrt{b + c}, a, \) and \( n \) (you may wish to start by finding new equations for the pythagorean triples involving \( q \) and \( n)). \) Is \( b + c \) always a perfect square? What condition on \( n \) and \( a \) is necessary for \( b + c \) to be a perfect square?